UGBA 198 Homework 1

February 7, 2018

1 Deriving Mode, Median, and Mean Under a Unifying Framework

Take a set of n numbers $x_1, x_2 ... x_n$. How do we summarize this data? In grade school, we've seen three possibilities: mode, median, and mean. As it turns out, we can interpret all three under one framework! The mode, median, and mean minimize the L0, L1, and L2 distances to all the numbers, respectively. More formally, they are the solutions to the following objective problem, for $m = 0, 1, 2$.

$$\min_u \sum_{i=1}^{n} |x_i - u|^m \tag{1}$$

To make this argument simpler, **define** $0^0 = 0$.

1. **Show that "mean" is the solution to (1) when $m = 2$.** Hint: Use what you know from calculus to minimize this objective.

$$\min_u \sum_{i=1}^{n} (x_i - u)^2$$

Solution: Take the derivative with respect to u.

$$\frac{\partial}{\partial u} \sum_{i=1}^{n} (x_i - u)^2 = \sum_{i=1}^{n} 2(x_i - u) = 0$$

Solving for u, we then get the following.

$$\begin{aligned}
\sum_{i=1}^{n} (x_i - u) &= 0 && \text{divide both sides by 2} \\
(\sum_{i=1}^{n} x_i) - (\sum_{i=1}^{n} u) &= 0 && \text{split sum} \\
\sum_{i=1}^{n} x_i - nu &= 0 && \text{u is constant with respect to i} \\
u &= \frac{1}{n} \sum_{i=1}^{n} x_i && \text{rearrange, solve}
\end{aligned}$$

2. **Show that "mode" is the solution to (1) when $m = 0$.** Hint: Use intuition.

$$\min_u \sum_{i=1}^{n} (x_i - u)^0$$

Solution: Examine when $x_i - u = 0$. We have $(x_i - u)^0 = 0^0 = 0$. Since we're minimizing the sum of all $(x_i - u)^0$, we want $x_i - u = 0$ more often. Equivalently, we want $x_i = u$ most often. In other words, we want the mode.

3. Start with two points–x_1 and x_2. **How do we choose u to minimize (1)?** Is there only one possible value or many? Hint: Consider two people standing on a line. Each person pays $1 to walk one foot. How do you pick a point for them to meet, so that they minimize the total cost they pay?

$$\min_u \sum_{i=1}^{n} |x_i - u|^1 = |x_1 - u| + |x_2 - u|$$

   Solution: Drawing two points on a line, we see that u can be any value between x_1 and x_2. Thus, u can be *anywhere* in the interval $[x_1, x_2]$.

4. Now, take three points–x_1, x_2 and x_3. **How do we choose u to minimize (1)?** Hint: Again, consider three people. Minimize the total distance all of them, collectively, need to walk.

   Solution: We must pick the point in the middle to be u.

   To reason about this, consider the two points on the outside, call them x_1 and x_3. Ignore x_2. We know that x_1 and x_3 contribute the same cost, *no matter where u is*, as long as u is between x_1 and x_3.

   Now, consider x_2. It contributes a non-zero amount if x_2 is not u. Since x_1, x_3 contribute constant cost regardless of where we place it, we can set $u = x_2$ to minimize cost for x_2.

5. Generally, how do we pick u to minimize (1) for an even number of points? For an odd number of points?

   Solution: For an even number of points, take the middle two points, and set u to be any value between those two. For an odd number of points, take the middle point. This is precisely a median.